Total Differential Equations

I. For three variables x, y, z the Pfaffian differential equation is of the form

$$P\,dx + Q\,dy + R\,dz \qquad \text{------} \quad (1)$$

Where P, Q, R are functions of x, y, z

This is also called a <u>total differential</u> equations in x, y, z. Such an equation can be directly integrated if $\exists$ a function $\phi(x,y,z)$ whose total differential

$$d\phi = P\,dx + Q\,dy + R\,dz.$$

If no such function $\phi(x,y,z)$ is available, then the equation (1) may or may not be integrable.

II. <u>The condition of Integrability</u>

<u>Theorem</u>:→ A Necessary and sufficient condition for integrability of the single differential equation

$$P\,dx + Q\,dy + R\,dz = 0 \qquad \text{------} \quad (1)$$

(Where P, Q, R are functions of x, y, z)

is that

$$P\cdot\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\cdot\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

Proof:→ Necessary condition

Suppose the equation ① have an integral

$$\phi(x,y,z) = c \quad \text{———} ②$$

$$\therefore \; d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$= P\,dx + Q\,dy + R\,dz$$

or equal to it multiplied by a factor $\mu(x,y,z)$, say

Thus

$$\frac{\frac{\partial \phi}{\partial x}}{P} = \frac{\frac{\partial \phi}{\partial y}}{Q} = \frac{\frac{\partial \phi}{\partial z}}{R} = \mu(x,y,z)$$

i.e; $\frac{\partial \phi}{\partial x} = \mu P, \; \frac{\partial \phi}{\partial y} = \mu Q, \; \frac{\partial \phi}{\partial z} = \mu R$ — ③

$$\therefore \; \frac{\partial}{\partial y}(\mu P) = \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial}{\partial x}(\mu Q)$$

$$\Rightarrow \mu \frac{\partial P}{\partial y} + P\frac{\partial \mu}{\partial y} = \mu \frac{\partial Q}{\partial x} + Q\frac{\partial \mu}{\partial x}$$

$$\Rightarrow \mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial y} \quad \text{———} ④$$

similarly, we easily obtain

$$\mu\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = R\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial z} \quad \text{———} ⑤$$

and $\mu\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P\frac{\partial \mu}{\partial z} - R\frac{\partial \mu}{\partial x}$ — ⑥

Multiplying ④, ⑤, ⑥ by R, P & Q respectively and adding, we obtain

$$\mu\left[R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)+P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)\right]=0$$

$$\Rightarrow P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

Sufficient Condition: →

Suppose that the coefficients P, Q, R in the equation

$$P\,dx + Q\,dy + R\,dz = 0 \quad \text{———} \quad (7)$$

satisfy the condition

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0 \quad \text{—} (8)$$

We shall prove that an integral of ① can be found when the relation ⑧ holds.

We first observe that if relation ⑧ holds for the coefficients of ①, then similar relation also holds for the coefficients of the relation:

$$\mu P\,dx + \mu Q\,dy + \mu R\,dz = 0 \quad \text{———} \quad (9)$$

Where μ is a function of (x, y, z).

consider the expression $Pdx + Qdy$ which is either an exact differential with respect to x & y or if it is not an exact differential, then an integrating factor μ can be found for it and in that case the equation

$$\mu P dx + \mu Q dy + \mu R dz = 0$$

Therefor, there is no loss of generality in regarding $Pdx + Qdy$ as an exact differential. For this the condition

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Let $V = \int Pdx + Qdy$, then it follows that

$$P = \frac{\partial V}{\partial x}, \quad Q = \frac{\partial V}{\partial y} \quad \left(\because dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy = Pdx + Qdy \right)$$

and $\dfrac{\partial P}{\partial z} = \dfrac{\partial^2 V}{\partial z \partial x}$, $\dfrac{\partial Q}{\partial z} = \dfrac{\partial^2 V}{\partial z \partial y}$

Hence from the condition ⑧, it follows that

$$\frac{\partial V}{\partial x}\left(\frac{\partial^2 V}{\partial z \partial y} - \frac{\partial R}{\partial y}\right) + \frac{\partial V}{\partial y}\left(\frac{\partial R}{\partial x} - \frac{\partial^2 V}{\partial z \partial x}\right) = 0$$

$$\left[\because R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = R\left(\frac{\partial^2 V}{\partial y \partial x} - \frac{\partial^2 V}{\partial x \partial y}\right) = R\left(\frac{\partial^2 V}{\partial y \partial x} - \frac{\partial^2 V}{\partial y \partial x}\right) = 0 \right]$$

This may be written as

$$\frac{\partial V}{\partial x} \cdot \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z} - R\right) - \frac{\partial V}{\partial y} \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z} - R\right) = 0$$

or the determinant

$$\begin{vmatrix} \frac{\partial V}{\partial x} & \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z} - R\right) \\ \frac{\partial V}{\partial y} & \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z} - R\right) \end{vmatrix} = 0$$

Which implies that a relation independent of x & y exists between

$$V \quad \text{and} \quad \frac{\partial V}{\partial z} - R$$

Therefor, $\frac{\partial V}{\partial z} - R$ can be expressed as a function of z and V alone.

Suppose that

$$\frac{\partial V}{\partial z} - R = \phi(z, V)$$

since

$$Pdx + Qdy + Rdz = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz + \left(R - \frac{\partial V}{\partial z}\right) dz$$

$\therefore$ The equation $Pdx + Qdy + Rdz = 0$ may be written as $dV - \phi(z, V)\, dz = 0$.

This is an equation in two variables. Its integration will lead to an equation of form

$$F(V,z) = 0$$

Hence the condition ⑧ is sufficient for ① to have an <u>integral</u>.

<u>Example ①</u> :→ Solve

$$(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2z\,dz = 0 \quad \text{— ①}$$

<u>Solution</u> :→

comparing the given equation with $P\,dx + Q\,dy + R\,dz = 0$,

We write

$$P = 2x^2 + 2xy + 2xz^2 + 1$$

$$Q = 1, \quad R = 2z$$

Now,

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial z}\right)$$

$$= (2x^2 + 2xy + 2xz^2 + 1)(0-0) + 1(0 - 4xz) + 2z(2x - 0)$$

$$= 0$$

Thus the condition of integrability is satisfied & hence the given equation is integrable.

Let x be treated as <u>constant</u>, so that $dx = 0$

$\therefore$ (1) becomes

$$dy + 2z\,dz = 0$$

Integrating,

$$y + z^2 = \text{constant} = f(x) \quad (\text{say}) \quad \text{— (2)}$$

Where the constant of integration has been taken as a function $f(x)$ of x because we have treated x as constant.

Differentiating (2), we get

$$dy + 2z\,dz - f'(x)\,dx = 0 \qquad \text{—— (3)}$$

Comparing (3) with the given equation (1), we have

~~dy + 2zdz~~ .

$$f'(x) = -\left[2x^2 + 2xy + 2xz^2 + 1\right]$$

or, $f'(x) = -(2x^2+1) - 2x(y+z^2)$

or, $f'(x) = -(2x^2+1) - 2x\,f(x)$

or, $\dfrac{d}{dx} f(x) + 2x\,f(x) = -(2x^2+1)$

This is a linear equation in $f(x)$ with

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

Hence multiplying by I.F., we get

$$\frac{d}{dx}\left\{f(x)\cdot e^{x^2}\right\} = -\int (2x^2+1)\,e^{x^2}\,dx$$

$$= -\int x\,d(e^{x^2}) - \int e^{x^2}\,dx$$

On integration

$$f(x)\cdot e^{x^2} = -xe^{x^2} + K \quad (K \text{ is an arbitrary constant})$$

i.e; $(f(x)+x)e^{x^2} = K$

or, $(y+z^2+x)e^{x^2} = K$

Ans.

<u>Example ②</u> Solve:

$$(y+z)dx + (z+x)dy + (x+y)dz = 0$$

<u>Solution:</u> → Here, $P = y+z$, $Q = z+x$ & $R = x+y$

$$\because \sum P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = 0$$

i,e; condition of integrability satisfied.

Rearranging the terms of the given equation,

$$(ydx + xdy) + (zdx + xdz) + (zdy + ydz) = 0$$

$$\Rightarrow d(x\cdot y) + d(z\cdot x) + d(zy) = 0$$

on integration

$$xy + yz + zx = K_1$$

Ans.

Example ③ Solve :

$$3x^2dx + 3y^2dy - (x^3+y^3+e^{2z})\,dz = 0 \quad \text{—} ①$$

Solution :→

comparing the given equation ① with

$$P\,dx + Q\,dy + R\,dz = 0$$

Then $P = 3x^2$, $Q = 3y^2$, $R = -x^3-y^3-e^{2z}$.

$$\because P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial z}\right)$$

$$= 3x^2(0+3y^2) + 3y^2(-3x^2-0) + (-x^3-y^3-e^{2z})(0-0)$$

$$= 0$$

so that the given equation is integrable.

Let z be treated as constant so that

$$dz = 0$$

Then ① becomes

$$3x^2dx + 3y^2dy = 0$$

Integrating, we have

$$x^3+y^3 = \text{constant} = f(z)$$

$$\Rightarrow 3x^2dx + 3y^2dy - f'(z)\,dz = 0$$

comparing it with the given equation ①

$$\therefore f'(z) = x^3+y^3+e^{2z} = f(z) + e^{2z}$$

$$\Rightarrow \frac{d}{dz}f(z) - f(z) = e^{2z}$$

This is a linear equation in $f(z)$.

$$\therefore \text{I.F.} = e^{-\int 1\,dz} = e^{-z}$$

Multiplying by I.F, we get

$$\frac{d}{dz}\{f(z)\cdot e^{-z}\} = e^{2z}\cdot e^{-z} = e^{z}$$

Integrating, we get

$$f(z)\cdot e^{-z} = e^{z} + k$$

(Where k is an arbitrary constant)

$$\Rightarrow (x^3+y^3)\,e^{-z} = e^{z} + k$$

or, $x^3+y^3 = e^{2z} + ke^{z}$

Ans.

Example ④ Solve

$$(x^2+y^2+z^2)\,dx - 2xy\,dy - 2xz\,dz = 0 \quad \text{—}(1)$$

Solution: →

comparing the given equation (1) with

$$P\,dx + Q\,dy + R\,dz = 0$$

We write

$$P = x^2+y^2+z^2,\quad Q = -2xy,\quad R = -2xz$$

Then $\sum P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = 0$

Let x be regarded as constant so that $dx=0$. Then the given equation becomes

$$-2xy\,dy - 2xz\,dz = 0$$

$$\Rightarrow 2y\,dy + 2z\,dz = 0$$

Integrating, we have

$$y^2+z^2 = \text{constant} = f(x),$$

Where $f(x)$ is taken as constant of integration as x treated as constant.

$$\Rightarrow 2y\,dy + 2z\,dz = f'(x)\,dx$$

comparing with the given equation (1)

$$\therefore x \cdot f'(x) = x^2+y^2+z^2$$

$$\Rightarrow f'(x) - \left(x + \frac{y^2+z^2}{x}\right) = 0$$

$$\Rightarrow f'(x) - \frac{f(x)}{x} = x$$

$$\Rightarrow \frac{d}{dx}\{f(x)\} - \frac{1}{x} f(x) = x$$

This is linear equation in $f(x)$

$$\therefore \text{I.F.} = e^{-\int \frac{1}{x}dx} = e^{\log\left(-\frac{1}{x}\right)} = \frac{1}{x}$$

Multiplying with I.F, we get

$$\frac{d}{dx}\left\{\frac{1}{x} f(x)\right\} = 1$$

on integrating, we get

$$\frac{1}{x} f(x) = x + K \quad (K \text{ is an arbitrary constant})$$

$$\Rightarrow \frac{1}{x}(y^2+z^2) = x + K$$

$$\Rightarrow y^2+z^2-x^2 = Kx$$

Ans.

Solution by Inspection

Example⑤ Solve $yz\log z\,dx - zx\log z\,dy + xy\,dz = 0$

Solution:→

$\because P = yz\log z\ ,\ Q = -zx\log z\ ,\ R = xy$

$$\because \sum P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = 0$$

i,e; condition of integrability is satisfied.

Now dividing throughout by $xyz\log z$ we get

$$\frac{dx}{x} - \frac{dy}{y} + \frac{\frac{dz}{z}}{\log z} = 0$$

Integrating

$$\log x - \log y + \log(\log z) = \log k \ \text{(say)}$$

i,e; $x\log z = ky$

Ans.

Total differential equation

Homogeneous Equations :→

The equation $Pdx + Qdy + Rdz = 0$ is called homogeneous equation if P, Q, R are all homogeneous function in x, y, z of same degree.

Working Rule :→

(1) If $D = Px + Qy + Rz \neq 0$, then $\frac{1}{D}$ as the integrating factor (I.F.)

(2) ($D = 0$ and the equation is homogeneous) put $x = zu$, $y = zv$ so that

$$dx = z\,du + u\,dz \quad \& \quad dy = z\,dv + v\,dz$$

substituting these values in the given equation we can solve the D.E.

Example (1) $(yz + z^2)dx - xz\,dy + xy\,dz = 0$

Solution:→ $\because$ ~~the~~ The given equation is homogeneous

& $P = yz + z^2$, $Q = -xz$, $R = xy$

$$\because D = Px + Qy + Rz$$

$$= x(yz + z^2) - xyz + xyz$$

$$= xz(y+z) \neq 0$$

$\therefore$ We take $\frac{1}{D} = \frac{1}{xz(y+z)}$ as the I.F.

Multiplying the given equation by I.F. we have

$$\frac{(yz+z^2)\,dx - xz\,dy + xy\,dz}{xz(y+z)} = 0 \quad — (1)$$

$$\therefore\ d(D) = d[xz(y+z)]$$

$$= z(y+z)\,dx + xz\,dy + 2xz\,dz + xy\,dz$$

$$= z(y+z)\,dx + xz\,dy + x(y+2z)\,dz$$

The Numerator of (1) becomes

$$z(y+z)\,dx - xz\,dy + xy\,dz$$

$$= \{z(y+z)\,dx + xz\,dy + x(y+2z)\,dz\} - 2xz\,dy - 2xz\,dz$$

$$= d(D) - 2xz\,d(y+z)$$

$\therefore$ Equation (1) becomes

$$\frac{d(D) - 2xz\,d(x+z)}{D} = 0$$

$$\Rightarrow \frac{d(D)}{D} - \frac{2xz\,d(y+z)}{D} = 0$$

$$\Rightarrow \frac{d(D)}{D} - \frac{2xz\,d(y+z)}{xz(y+z)} = 0$$

$$\Rightarrow \frac{d(D)}{D} - 2\,\frac{d(y+z)}{(y+z)} = 0$$

Integrating

$$\log D - 2\log(y+z) = \log c \quad \text{(c is a constant)}$$

$$\Rightarrow \quad \frac{D}{(y+z)^2} = c$$

$$\Rightarrow \quad \frac{xz(y+z)}{(y+z)^2} = c$$

$\therefore \quad xz = c(y+z)$ is the required solution.

Example ②: → Solve

$$yz(y+z)dx + zx(x+z)dy + xy(x+y)dz = 0 \quad \text{— ①}$$

Solⁿ: → $\because$ The given equation satisfies the condition of integrability.

$\because$ The equation is homogeneous.

Let $x = uz, \quad y = vz$

$\therefore \; dx = u\,dz + z\,du \quad \& \quad dy = v\,dz + z\,dv$

putting these values in equation ①

$$[v(v+1)du + u(u+1)dv]\,z^4 + 2uv(u+v+1)z^3 dz = 0$$

Dividing by $uv(u+v+1)z^4$, we get

$$\frac{(v+1)du}{u(u+v+1)} + \frac{(u+1)\,dv}{v(u+v+1)} + \frac{2\,dz}{z} = 0$$

$$\Rightarrow \left(\frac{1}{u} - \frac{1}{u+v+1}\right)du + \left(\frac{1}{v} - \frac{1}{u+v+1}\right)dv + \frac{2dz}{z} = 0$$

on integrating

$$\log u + \log v + 2\log z - \frac{1}{u+v+1}\,d(u+v+1) = \log c$$

(constant)

$$\Rightarrow \frac{uvz^2}{u+v+1} = c \quad \text{i.e.;} \quad \frac{\frac{x}{z}\cdot\frac{y}{z}\cdot z^2}{\frac{x}{z}+\frac{y}{z}+1} = c$$

i.e.; $\quad xyz = c(x+y+z)$ (Required solution)